TENSOR CHARACTERISTICS OF COHERENT VORTEX STRUCTURES

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Elements of chaos, as well as order, are observed in turbulent flows. Partial order in turbulent flows is manifested through the presence of coherent (organized) structures. The identification of organized motions in turbulent flow is made difficult because such motions evolve, and the phase as well as orientation are random. The main difficulty, however, is that since the interaction of structures is sufficiently strong, concrete realizations fluctuate significantly.

Flow visualization techniques to study the structures are reliable when they are isolated. If the orientation and phase of the structures are fixed, then the quantitative information is obtained by conditional averaging of the hot-wire signal [1-3]. In the opposite case, single-point measurements do not give the desired information on organized structures.

In recent years, simultaneous measurements at many points in the turbulent flow field have been carried out [4-6]. Methods are being developed to numerically simulate turbulence on the computer. It is clear that the complete information on the organized structures is present in the instantaneous hydrodynamic flow field. But the question arises as to how to retrieve this information.

A complete understanding of the structures under conditions of large fluctuations demands a statistical approach. For statistical study it is necessary to develop a system of quantitative characteristics; these should describe structures of given scale in the neighborhood of each point in the flow.

The present work describes a method to analyze results of laboratory and numerical experiments for the identification and quantitative analysis of successive three- and two-dimensional organized structures, and requires the computation of tensor moments of the flow field. The tensor field developed below contains information on the fundamental characteristics of the flow in the neighborhood of each point and the invariants of rotation do not depend on the orientation of the ordered motion and could be used to determine its qualitative characteristics. One could hope that the statistical properties of invariants would help in understanding what could be reproduced and what fluctuates in turbulent flows.

Identification of Vortex Structures of a Given Scale. Coherent structures refer to coherent bundles of vorticity. In order to isolate an organized structure of a given scale λ, it is necessary to have a proper description of the vortex field. Consider a spherical volume of radius λ in a turbulent fluid. The boundary of this volume could be intersected by vortex lines which cannot be broken without violating the irrotationality condition. Divide the vorticity ω into two solenoidal components ω' and $\omega^{(\lambda)}$. It is assumed that the ω' field coincides with ω outside the chosen volume and is equal to the gradient of the harmonic function $\nabla\chi$ inside the volume. The condition of irrotationality for ω' and $\omega^{(\lambda)}$ is satisfied everywhere if at the boundary of the region the normal derivative of the function χ is equal to the normal projection of vorticity:

$$(\mathbf{r}\cdot\nabla\chi)_\Gamma = (\mathbf{r}\cdot\boldsymbol{\omega})_\Gamma. \tag{1}$$

Here r denotes coordinates of any point relative to the center of the volume; the index Γ refers to computation at the spherical boundary. The ω'-field does not contain information on ω inside the volume; this is present in the additional component $\omega^{(\lambda)} = \omega - \omega'$. The $\omega^{(\lambda)}$-field forms a vortex of size λ since its vortex lines do not intersect the boundary of the volume. Thus, the description of the vortex structure inside the chosen volume leads to the problem of describing the structure of an isolated vortex.

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Moments of Vortex Structure. The distribution of vorticity inside the structure can be very complex. Not all details of this distribution are of equal interest and importance, and hence it is desirable to express integral parameters which characterize basic properties of the distribution in its entirety.

The known integral characteristics of the vortex cloud are the vortex momentum and the angular moment of vortex momentum (Lamb's momentum) [7, 8]:

$$\mathbf{P} = \frac{1}{2}\int \mathbf{r} \times \boldsymbol{\omega}^{(\lambda)} dV, \quad \mathbf{J} = \frac{1}{3}\int \mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega}^{(\lambda)})\, dV$$

(for an isolated vortex it is possible to put $\omega^{(\lambda)} \equiv \omega$). More subtle details of vortex distribution are described by linear moments

$$M^{(n)}_{i_1\ldots i_n}(\mathbf{x}) = \int r_{i_1} r_{i_2} \ldots r_{i_{n-1}} \omega^{(\lambda)}_{i_n}(\mathbf{x}+\mathbf{r})\, dV(\mathbf{r}), \tag{2}$$

where $\mathbf{x}$ are coordinates of the center of the sphere. As mentioned in [9], moments of the type (2) are convenient to describe internal degrees of freedom of isolated vortices. In the present case $M^{(n)}$ is the tensor field characterizing the flow structure in the neighborhood of each point $\mathbf{x}$. An analogous method was used in [10] to describe the local structure of condensed matter.

It is not difficult to note that the moments $M^{(n)}$ are the coefficients of Taylor series of the Fourier transform:

$$\tilde{\omega}_m(\mathbf{k}) = \int \exp(-i\mathbf{k}\mathbf{r})\, \omega^{(\lambda)}_m(\mathbf{r})\, dV(\mathbf{r}).$$

Hence the $\omega^{(\lambda)}$ field is completely described by the set of moments (2). The most significant characteristics of the chosen vortex are given by moments of lower order. Consider in great detail M^n ($n \leq 4$). It is convenient to divide these tensors into irreducible components [11]. Irreducible tensor components $M^{(n)}$ ($n \leq 4$) expressed in terms of vorticity ω are given by

$$\mathbf{P} = \frac{1}{2}\int \mathbf{r} \times \boldsymbol{\omega} dV; \tag{3}$$

$$J^{*}_{i} = \frac{1}{5}\int (3r_i r_j - r^2\delta_{ij})\, \omega_j dV; \tag{4}$$

$$t_{ij} = \int [r_i(\mathbf{r}\times\boldsymbol{\omega})_j + r_j(\mathbf{r}\times\boldsymbol{\omega})_i]\, dV; \tag{5}$$

$$d_{ij} = \frac{4}{7}\int [5r_i r_j r_m - r^2(r_j\delta_{im} + r_m\delta_{ij} + r_i\delta_{jm})]\, \omega_m dV; \tag{6}$$

$$\mathbf{a} = \frac{1}{10}\int r^2\mathbf{r} \times \boldsymbol{\omega} dV; \tag{7}$$

$$c_{ijm} = \frac{1}{3}\int [r_i r_j(\mathbf{r}\times\boldsymbol{\omega})_m + r_i r_m(\mathbf{r}\times\boldsymbol{\omega})_j + r_j r_m(\mathbf{r}\times\boldsymbol{\omega})_i]\, dV - \frac{2}{3}(a_i\delta_{jm} + a_j\delta_{im} + a_m\delta_{ij}). \tag{8}$$

Here the integrals are taken about the chosen spherical volume. Transformations to integrals from the ω field are carried out using Green's function for the Neumann problem [12]. Some details of computation are given in [13]. Vorticity of the type $\omega' = \nabla\chi$ and a simple transfer by large-scale motions do not contribute to tensors (3)-(8).

The physical meaning of irreducible tensors becomes clear if they are expressed in terms of the density of vortex momentum q [14, 15]. The q field is determined such that it is equal to the density of momentum required for the instantaneous generation of vortex $\omega^{(\lambda)}$ in the background of vorticity ω'. The substitution of $\mathbf{f} = \delta(t)\mathbf{q}$ in the vorticity equation

$$\partial\boldsymbol{\omega}/\partial t = \mathrm{rot}\,[\mathbf{u}\times\boldsymbol{\omega}] + \nu\Delta\boldsymbol{\omega} + \mathrm{rot}\,\mathbf{f}$$

and the integration along infinitely small time interval $(-\varepsilon, \varepsilon)$, $\varepsilon \to 0$, gives the relation $\omega^{(\lambda)} = \operatorname{rot} \mathbf{q}$. The calibration of the **q** field is chosen such that the isolated vortex represented the spatially bound distribution $\mathbf{q}(\mathbf{r})$. Other details of calibration are of no further significance. Due to the finiteness of the **q** field, its moments could be computed by integrating about the entire space.

The substitution of $\omega = \nabla\chi + \operatorname{rot} \mathbf{q}$ in (3)-(8) and the integration by parts result in

$$\mathbf{P} = \int \mathbf{q} dV; \tag{9}$$

$$\mathbf{J} = \int \mathbf{r} \times \mathbf{q} dV; \tag{10}$$

$$t_{ij} = \int [3(r_i q_j + r_j q_i) - 2(\mathbf{r} \cdot \mathbf{q}) \delta_{ij}] dV; \tag{11}$$

$$d_{ij} = 4 \int [r_i (\mathbf{r} \times \mathbf{q})_j + r_j (\mathbf{r} \times \mathbf{q})_i] dV; \tag{12}$$

$$\mathbf{a} = \frac{1}{5} \int [2\mathbf{q} r^2 - \mathbf{r} (\mathbf{r} \cdot \mathbf{q})] dV; \tag{13}$$

$$c_{ijm} = \frac{4}{15} \int [5(r_i r_j q_m + r_i r_m q_j + r_j r_m q_i) - 2(\mathbf{r} \cdot \mathbf{q})(r_i \delta_{jm} + r_j \delta_{im} + r_m \delta_{ij}) - r^2 (q_i \delta_{jm} + q_j \delta_{im} + q_m \delta_{ij})] dV. \tag{14}$$

Equations (9)-(14) make it possible to express the nature of the motion through moments of the distribution of equivalent force momentum, and Eqs. (9) and (10) lead to the known conclusion [7, 8] that vectors **P**, **J** are equal to the total vortex momentum and the moment of vortex momentum. Tensors t_{ij} and c_{ijm} describe quadruple and octuple deformation of the liquid volume by vorticity $\omega^{(\lambda)}$. The tensor d_{ij} gives distortion along the major axes of this tensor. The vector **a** describes recirculation inside the chosen volume associated with $\omega^{(\lambda)}$.

Smoothed Section. The choice of the most convenient representation of moments is dictated by the specific nature of the problem. Moments (3)-(8) can be expressed in terms of velocity using kinematic conditions $\omega = \operatorname{rot} \mathbf{u}$ and integrating by parts. This results in surface integrals of velocity on the sphere, which could be undesirable. For example, let the velocity be measured at certain finite number of points. Statistically some of them may be on the spherical boundary, but may be insufficient for a reliable computation of the surface integral. It is possible to avoid these difficulties by excluding integration along the region with the given boundary $r = \lambda$.

Introducing a weighting function $w^{(\lambda)}(r)$ for the integrals (3)-(8) and extending the integration to the entire space, e.g.,

$$\mathbf{P} = \frac{1}{2} \int \mathbf{r} \times \omega(\mathbf{x} + \mathbf{r}) w^{(\lambda)}(r) dV(\mathbf{r}).$$

It is assumed that the function $w^{(\lambda)}(r)$ is close to unity when $r < \lambda$ and rapidly decreases outside this region. The above integral corresponds to the stepwise discontinuous region $w^{(\lambda)}(r) = \theta(\lambda - r)$, where θ equals one for $r < \lambda$ and zero for $r > \lambda$. The case of an arbitrary function $w^{(\lambda)}(r)$ leads to the above case if a step function is used in the integral $w^{(\lambda)}$

$$w^{(\lambda)}(r) = -\int_0^\infty \frac{dw^{(\lambda)}(\lambda')}{d\lambda'} \theta(\lambda' - r) d\lambda'. \tag{15}$$

Equation (15) makes it possible to separate an isolate vortex corresponding to the smooth $w^{(\lambda)}$. Direct computations show that its vorticity is solenoidal and equals

$$\omega^{(\lambda)}(\mathbf{r}) = -\int_r^\infty d\lambda' \frac{dw^{(\lambda)}(\lambda')}{d\lambda'} [\omega(\mathbf{r}) - \nabla\chi(\mathbf{r}, \lambda')],$$

where $\chi(r, \lambda')$ is a harmonic function satisfying the boundary condition (1) at $r = \lambda'$. Vortex parameters are determined from the integrals (3)-(8) with the replacement $dV \to w^{(\lambda)} dV$.

Surface integrals of velocity are transformed to volume integrals of the spherical layer whose effective thickness is determined by the profile of the function $dw^{(\lambda)}(r)/dr$.

Two-dimensional vortex structures are described by simpler tensor characteristics since vorticity can also be considered pseudoscalar. The above fluctuation ω' is supplemented by the vortex component that satisfies the condition $\mathrm{rot}\,\omega' = 0$ inside the chosen region. In the case of two-dimensional flows, this condition has the form $e_{ij}\partial\omega'/\partial r_j = 0$, i.e., $\omega' = \mathrm{const}$.

Consider a circle of radius λ in the plane of the flow, and divide vorticity into two components: $\omega = \omega' + \omega^{(\lambda)}$. The ω' field coincides with ω outside the chosen region and is constant inside it. It is convenient to make this constant equal to the mean velocity along the circle

$$\omega'(\mathbf{r}) = \frac{1}{\pi\lambda^2}\int \omega dA \quad (r < \lambda).$$

The field $\omega^{(\lambda)} = \omega - \omega'$ forms a vortex pair of scale λ. Its moments are symmetrical relative to the rotation of all indices:

$$M^{(n)}_{i_1 \ldots i_{n-1}}(\mathbf{x}) = \int r_{i_1} \ldots r_{i_{n-1}} \omega^{(\lambda)}(\mathbf{x}+\mathbf{r})\, dA(\mathbf{r}).$$

From $M^{(n)}$ it is possible to construct irreducible moments of vortex pairs. The simplest of them, viz., vortex impulse, moment of momentum, and tensor deformation:

$$P_i = e_{ij}\int r_j \omega dA, \quad J = \frac{1}{2}\int \left(r^2 - \frac{\lambda^2}{2}\right)\omega dA,$$
$$t_{ij} = \int \left(r_i r_j - \frac{1}{2} r^2 \delta_{ij}\right)\omega dA.$$

These moments are not affected in the transformation to a rotating system of coordinates or in other Galilean systems which are important for possible geophysical applications.

Invariance of Tensors. The tensor field described in this work is nonzero only near the points where structures of scale λ are present. This property is useful in the study of flows with noticeable displacement. Tensors (3)-(8) can be used to bring out organized structures and also for their quantitative analysis. An invariant symmetric tensor of rank n contains 2n + 1 independent components. Three parameters give the orientation of the structure in space. The remaining 2(n − 1) parameters give quantitative characteristics of the vortex structure independent of its orientation. These characteristics can be considered invariant $\psi^{(\mu)}$ (μ = 1, 2, ...) of irreducible tensors.

Assume that tensor fields $M^{(n)}$ and their invariants for $n \leq n_0$ are obtained from experiment or by numerical simulation. Questions arise as to which information on the nature of turbulent structures can thus be obtained and how to get it.

Knowing the distribution of invariants inside the volume of the system, it is possible to isolate parts of the fluid occupied by corresponding structures. It is possible to identify the type of structures across their fields $\psi^{(\mu)}(\mathbf{x})$ by comparing these fields with a standard based on specially created vortex-disturbance types (vortex rings and vortex filaments, shear layers, etc.), and in doing so it becomes possible to set up a spectroscopy of structures according to their invariants. It is especially interesting to compare the evolution of the quantities $\psi^{(\mu)}$ in time with the evolution of isolated structures. A comparison of structures according to their moments $M^{(n)}$ for $n \leq n_0$ is the comparison of classes to which these structures belong. It is possible to expect that invariants $\psi^{(\mu)}$ for small numbers ($n \leq 4$) describing the most large-scale deformation of the volume in flow are not too sensitive to fluctuations inside the structures.

Identification of vortex structures in turbulent flow with standard configuration indicates that, for this structure, its characteristics are expressed as the projection of fluctuations on the characteristics of the standout structure. The choice of initial time and location and orientation in space still remain arbitrary in such a comparison. It is possible that stochastic nature of turbulence is primarily associated with these quantities and hence their study in terms of moments $M^{(n)}$ is an important problem.

LITERATURE CITED

1. A. A. Townsend, The Structure of Turbulent Shear Flow, Cambridge Univ. Press (1976).
2. B. J. Cantwell, "Organized motion in turbulent flow," Ann. Rev. Fluid Mechanics, 13 (1981).
3. A. K. M. F. Hussain, "Coherent structures – reality and myth," Phys. Fluids, 26, No. 10 (1983).
4. R. R. Johnson, E. Push, et al., "Experiments on the structure of turbulent shear in pipe flow of water," Phys. Fluids, 19, No. 9 (1976).
5. V. V. Orlov, E. S. Mikhailova, and É. M. Khabakhpasheva, "Semiautomatic measurement of kinematic characteristics in turbulent flows of liquids and gases," Metrologiya, No. 3 (1970).
6. B. G. Novikov, V. D. Fedosenko, et al., "Stereometric study of farfield free shear flow turbulence," in: Hydrodynamics and Acoustics of Wall Layers and Free Flows, B. P. Mironov (ed.) [in Russian], Inst. Teplofiziki, Siberian Branch, Academy of Sciences of the USSR, Novosibirsk (1981).
7. H. Lamb, Hydrodynamics, Cambridge (1957).
8. J. Batchelor, An Introduction to Fluid Dynamics, Cambridge Univ. Press (1970).
9. P. H. Roberts, "A Hamiltonian theory for weakly interacting vortices," Matematika, 19, No. 1 (1972).
10. A. Z. Patashinskii, Structure of Condensed State and Phase Transition in Amorphous Systems [in Russian], Inst. Yadernoi Fiziki, Siberian Branch, Academy of Sciences of the USSR, No. 64 (1984).
11. I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, Rotational Groups and Lorentz Groups [in Russian], Fizmatgiz, Moscow (1958).
12. N. S. Koshlyakov, É. B. Gliner, and M. M. Smirnov, Basic Differential Equations of Mathematical Physics [in Russian], Fizmatgiz, Moscow (1962).
13. G. A. Kuz'min and A.Z. Patashinskii, Parameters of Organized Structures of Turbulent Flows [in Russian], Inst. Yadernoi Fiziki, Siberian Branch, Academy of Sciences of the USSR, No. 155 (1984).
14. G. A. Kuz'min, "Ideal incompressible hydrodynamics in terms of the vortex momentum density," Phys. Lett. A., 96, No. 2 (1983).
15. G. A. Kuz'min, "Hydrodynamics of stratified fluid in terms of Lamb's momentum density," Prikl. Matem. Tekh. Fiz., No. 4 (1984).